BINOMIAL OPTION PRICING

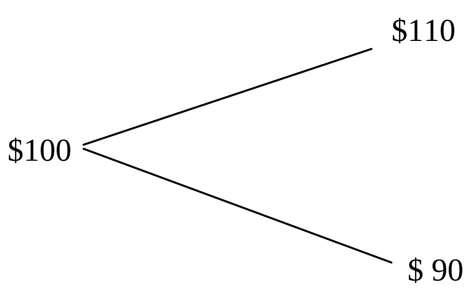
Binomial option pricing is a simple but powerful technique that can be used to solve many complex option-pricing problems. In contrast to the Black-Scholes and other complex option-pricing models that require solutions to stochastic differential equations, the binomial option-pricing model (two-state option-pricing model) is mathematically simple. It is based on the assumption of no arbitrage.

The assumption of no arbitrage implies that all risk-free investments earn the risk-free rate of return and no investment opportunities exist that require zero dollars of investment but yield positive returns. It is the activity of many individuals operating within the context of financial markets that, in fact, upholds these conditions. The activities of arbitrageurs or speculators are often maligned in the media, but their activities insure that our financial markets work. They insure that financial assets such as options are priced within a narrow tolerance of their theoretical values.

BINOMIAL OPTION-PRICING MODEL

Assume that we have a share of stock whose current price is $100/share. During the next month, the price of the stock is either going to go up to $110 (up state) or go down to $90 (down state). No other outcomes are possible over the next month for this stock's price.

<u>Beginning Value</u> **<u>End of Month Value</u>**



____

This note was prepared by Professor Robert M. Conroy.

Now assume that a call option exists on this stock. The call option has a strike price of $100 and matures at the end of the month. The value of this call option at the end of the month will be $10 if the stock price is $110 and 0 if the stock price is $90. The payoff at maturity (one month from now) for this call option is:

Beginning Value		**End of Month Value**
		$10 given a stock price of $100
Call Price today		
		$ 0 given a stock price of $ 90

The question is: what should be the price of the call option today?

Consider what happens when we make the following investments in the stock and the call option. Assume we buy one-half share of stock at $50 (.5 share times $100), and at the same time, we write one call option with a strike price of $100 and maturing at the end of the month. Our investment then is $50 less the current price of the call option. The payoff from this position at the end of the month would be as follows: if the stock price is $110, our stock position is worth $55, and we would lose $10 on the option. The return would thus be $45 if the stock price reached a price of $110. On the other hand, if the stock price should go down to $90, the value of our stock position would be $45 and the value of our option position would be 0. The payoff in this case would also be $45.

Investment		**Payoff**
		1/2 share of stock - call option $55 - Max[110-100,0] = $45
1/2 share of stock - call option $50 - call option		
		1/2 share of stock - call option $45 - Max[90-100,0] = $45

The net effect of taking this particular position on this stock with this payoff structure is that our payoff is $45 regardless of what happens to the stock price at the end of the month. The effect of buying 1/2 a unit of the stock and writing a call option was to change a risky position into one that is risk free with a payoff of $45 regardless of the stock price at the end of the month. Assuming no arbitrage opportunities, an investor who makes this investment should earn exactly the risk-free rate of return. Thus, we know that the investment, $50 minus the call option price, has to be equal to the present value of $45, the payoff, discounted for 1 month at the current risk-free rate of return.

In order to find the current price of the call option, we need only solve the following equation for the option price:

$$\$50 \ - \ Option\ price \ = \ \$45 \cdot e^{-R_F \cdot T}$$

$$Option\ price \ = \ \$50 \ - \ \$45 \cdot e^{-R_F \cdot T}$$

where R_F is the risk-free rate and T is the time to maturity in years. Assuming that the current risk-free rate of return is 6% per annum and a time to maturity of one month, T=.08333, the current option price of this call option should be $5.22.

The process used to price the option in this example is exactly the same procedure or concept used to price all options, whether with the simple binomial option model or the more complicated Black-Scholes model. The assumption is that we find and form a risk-free hedge and then price the option off of that risk-free hedge. The key assumption is that the risk less hedge will be priced in such a way that it earns exactly the risk-free rate of return, which is where arbitrageurs come in to play. It is the activity of these individuals, looking for opportunities to invest in a risk less asset and earn more than the risk-free rate of return that insures that options are priced according to the no-arbitrage conditions.

In general, there are two approaches to using the binomial model, the Risk-less Hedge Approach and the Risk-Neutral Approach. Either approach will yield the same answer, but the underlying approach differs. This note examines each approach.

Riskless Hedge Approach

Estimating the risk less hedge

In the example, the hedge ratio, or the number of units of the stock held per call option was given. The next step is to demonstrate how we would find the appropriate hedge ratio for a stock. Assume the following payoff structure for a stock over the next month.

<u>Beginning Value</u> **<u>End of Month Value</u>**

$95

$ 75

$63

In this case, the current stock price is $75, and at the end of the month, the stock will be either $95 or $63. Suppose we also had a call option with a strike price of $65. The payoff for this call option at the end of the month is

<u>Today</u> **<u>Payoff</u>**

$30 given a stock price of $95 (Max[95-65,0])

Call Price today
Exercise price=$65

$ 0 given a stock price of $63 (Max[63-65,0])

Let H represent the number of units of the stock we should hold. The investment would be H x $75 - call option price. If the stock price were $95 at the end of the month, the value of the call options would be $30. The payoff on our total investment would be H x $95 - $30. If the stock price at the end of the month were $63, the value of the call option would be $0. The payoff from the total position on the stock and call option would be H x $63 - $0.

<u>Investment</u> **<u>Payoff</u>**

H shares of stock - call option
H x $95 - Max[95-65,0]

H share of stock - call option

H shares of stock - call option
H x $63 - Max[63-65,0]

We find the appropriate risk-free hedge, H, by setting the payoff in the up state equal to the payoff in the down state, or

$$H \text{ x } \$95 - \$30 = H \text{ x } \$63 - 0.$$

$$H = \frac{\$30 - \$0}{\$95 - \$63} = .9375.$$

Solving for H[1], we find that the payoffs would be exactly the same in each state, if H is equal to .937. For H = .9375, the payoff in the up state is .9375 x \$95 - 30, or \$59.0625. In the down state .9375 x \$63 - 0, or \$59.0625. Thus, by buying .9375 shares of the stock and writing a call option, we have created a riskless hedge with a payoff of \$59.0625 regardless of what the stock price is at the end of the month. Assuming no arbitrage opportunities, the investment, .9375 x \$75 - C, where C is the call price, has to be equal to the present value of the payoff of \$59.0625 discounted at the risk-free rate of return. Again, assuming a risk-free rate of return of 6% per annum and time to maturity of one month, T=.08333, we find that the call price should be equal to \$11.54.

$$.9375 \text{ x } \$75 - C = \$59.0625 \text{ x } e^{-R_F \cdot T}$$

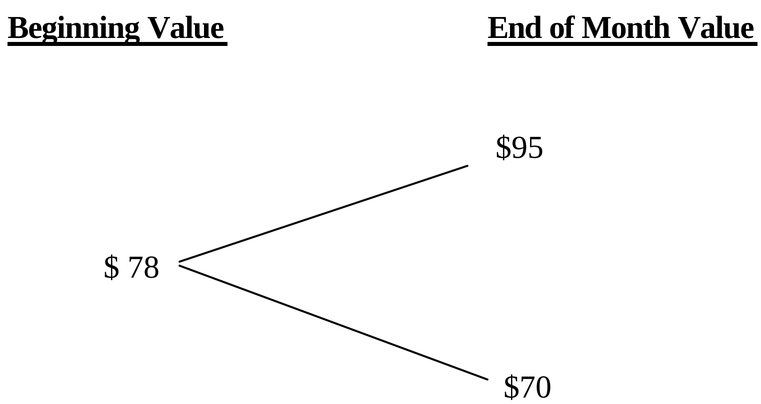
$$\$70.3125 - C = \$59.0625 \text{ x } e^{-.06 \cdot .08333}$$

$$C = \$70.3125 - \$59.0625 \text{ x } e^{-.06 \cdot .08333}$$

$$C = \$\ 11.54$$

[1] If we look at the estimate of the hedge ratio, it is the ratio of the change in the call option price to the change in the stock price. More formally, this ratio is an estimate of the rate of change in the value of the option relative to the change in the stock price. It measures the change in option value per \$1 dollar change in stock price. In a more formal framework, this is the Delta of the call option. If we hold H (H = Delta) units of stock and write one call option, a \$1 dollar change in the stock price results in a H x \$1 increase in the value of our stock position, and this is offset by the H x \$1 change in the value of our option position. Effectively our total position is “Delta neutral”.

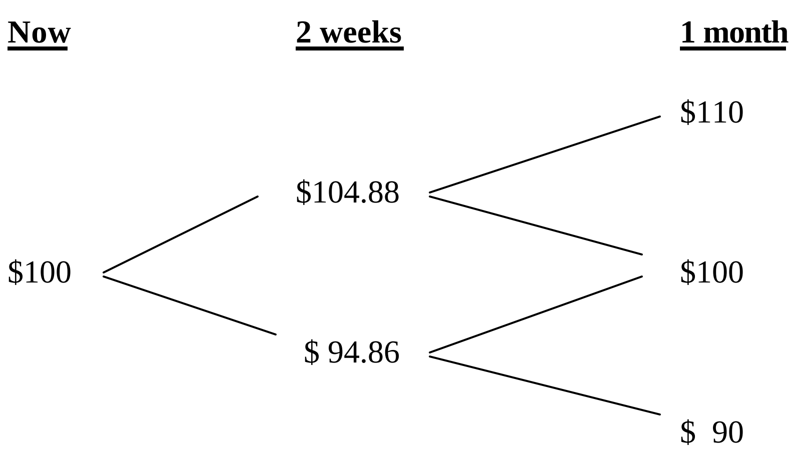
Self-Test Example: Price an option with the strike price of $80 that matures at the end of one month for the stock with the end-of-month price distribution shown in below.



Assuming a risk-free rate of 6% per annum, you should get a hedge ratio of .60 and an option price of $5.01.

USING THE BINOMIAL OPTION-PRICING MODEL FOR MORE THAN ONE PERIOD

Suppose we were to take the original example, and instead of having only a single price change during the month, let us assume that the price change is once every 2 weeks. You can see that, by dividing the month up into 2 periods, we wind up with 3 possible outcomes at the end of the month.

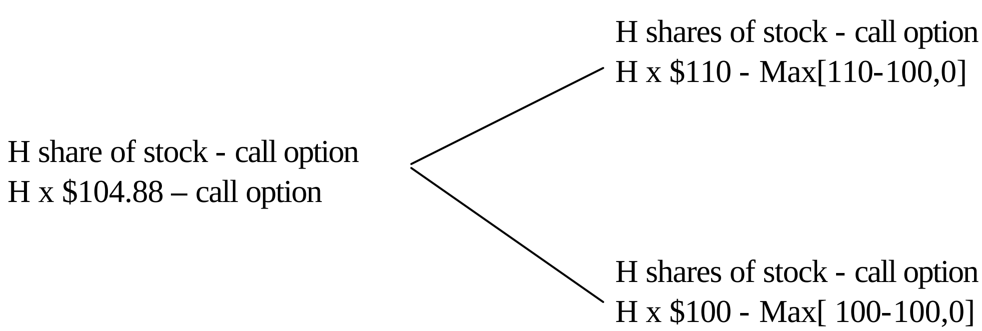


Using this pricing dynamic, what would be the price of an option today that matures at the end of the month and has a strike price of $100?

The approach to solving this problem is really no more difficult than the original one. The strategy for solving this multi-period problem is to break it up into a number of simple two-period models. We work backwards!

Consider the branch where the stock price reaches $104.88. From there, during the next 2-week period, the stock price will either increase to $110 or decrease to $100. So we can ask at this point: assuming that the stock price is $104.88, what is the option worth with just 2 weeks to go? Once again, the first step is to assume that we buy H units of the stock with 2 weeks to go and write one call option. The number of units of the stock we need to buy is the risk-free hedge. Thus, as before, we determine the payoff structure. This structure is shown as follows

Investment **Payoff**



We need to choose H such that H x $110 - 10 = H x $100 - 0. The value of H that solves this equality is 1.0 and if H=1.00 the payoff is always $100. Thus, if the stock price is $104.88 and we buy one unit of the stock and write one call option, our payoff would be $100 regardless of what the stock price is at the end of the 2-week period. Assuming a risk-free rate of 6% per annum and time to maturity of 2 weeks (T=.04167), the value of the call option when the stock price is $104.88 is $5.13.

$$1.000 \text{ x } \$104.88 - C = \$100.000 \text{ x } e^{-R_F \cdot T}$$

$$\$104.88 - C = \$100.00 \text{ x } e^{-.06 \cdot .04167}$$

$$C = \$104.88 - \$100.00 \text{ x } e^{-.06 \cdot .04167}$$

$$C = \$\ 5.13$$

If the stock price at the end of the first 2-week period is $94.86, then for the subsequent 2-week period, the stock price is either going to go up to $100 or down to $90. In either case, the call option will be worthless at the end of that time period, because no one would ever choose to exercise the call options with a strike price of $100. At the end of the first two weeks, therefore, if the stock price goes to $94.86, the price of the option would be $0.

We now move back to the first 2-week period. The payoff structure from buying the stock and writing H call options is shown below:

Investment	**Payoff**
	H shares of stock - call option H x $104.88 - $ 5.13
H shares of stock - call option	
	H shares of stock - call option H x $ 94.86 - $ 0

Note that a new hedge ratio is needed, but the problem is exactly the same as the simple problem we faced originally. We first find the value of the H that leads to a riskless payoff. H in this case would have to be equal to .51198.

$$H \times \$104.88 - \$5.13 = H \times \$94.86 - \$0.$$

$$H = \frac{\$5.13 - \$0}{\$104.88 - \$94.86}$$

$$H = .51198$$

In this case we buy .51198 units of the stock and write one call option. Our payoff would be exactly $48.566 regardless of what happened to the stock price[2]. The value of the call options would

2

.51198 x $104.88 - 5.13 = $48.566
.51198 x $ 94.86 - 0 = $48.566

have to satisfy the following equation:

$$.51198 \cdot \mathit{\$100} - \mathit{Option\ price} = \$48.566 \cdot e^{-R_F \cdot T}$$

Solving for the call option price, we find that it would have to be equal to \$2.753.

$$\mathit{Option\ price} \ = \ \$51.198 - \$48.56 \cdot e^{-.06 \cdot .04167}$$

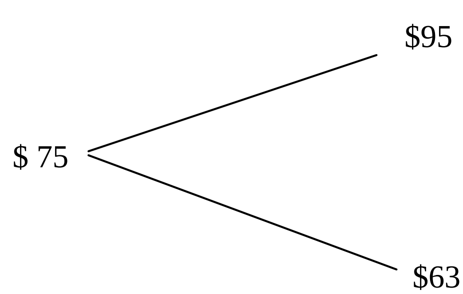
$$\mathit{Option\ price} = \$2.753$$

Risk Neutral Approach

An alternative to the risk less hedge approach to valuing options using the binomial model is the risk neutral approach. The basic argument in the risk neutral approach is that since the valuation of options is based on arbitrage and is therefore independent of risk preferences; one should be able to value options assuming any set of risk preferences and get the same answer. As such, the easiest model is the risk neutral model.

In the risk less hedge approach, the probability of the stock price increasing, P_u, or the probability of the stock price decreasing, $P_d = 1\text{-}P_u$, did not enter into the analysis at all. In the risk neutral approach, given a stock price process (tree) we try to estimate these probabilities for a risk neutral individual and then use these risk neutral probabilities to price a call option. For example, we will use the same price process as the original risk less hedge example.

Beginning Value **End of Month Value**



If an individual is risk neutral, then they should be indifferent to risk and as such for them the current stock price is the expected payoff discounted at the riskfree rate of interest. Assuming a 6% riskfree rate, a risk neutral individual would make the following assessment:

$$\$75 = \left[P_u \cdot \$95 + \left(1 - P_u\right) \cdot \$63\right] \cdot e^{-R_F \cdot T}$$

If we solve for P_u,

$$P_u = \frac{\$75 \cdot e^{R_F \cdot T} - \$63}{\$95 - \$63}.$$

If R_F = 6% and T=.08333, then P_u = .38675. This is the risk neutral probability of the stock price increasing to $95 at the end of the month. The probability of it going down to $63 is 1-.38675= .61325. Now given that if the stock price goes up to $95, a call option with an exercise price of $65 will have a payoff of $30 and $ 0 if the stock price goes to $63, a risk neutral individual would assess a .38657 probability of receiving $30 and a .61325 probability of receiving $ 0 from owning the call option. As such, the risk neutral value would be:

$$\text{Call Option Value} = \left[P_u \cdot \$30 + \left(1 - P_u\right) \cdot \$0\right] \cdot e^{-.06 \cdot .08333}$$

$$\text{Call Option Value} = \left[.38657 \cdot \$30\right] \cdot e^{-.06 \cdot .08333}$$

$$\text{Call Option Value} = \$11.54$$

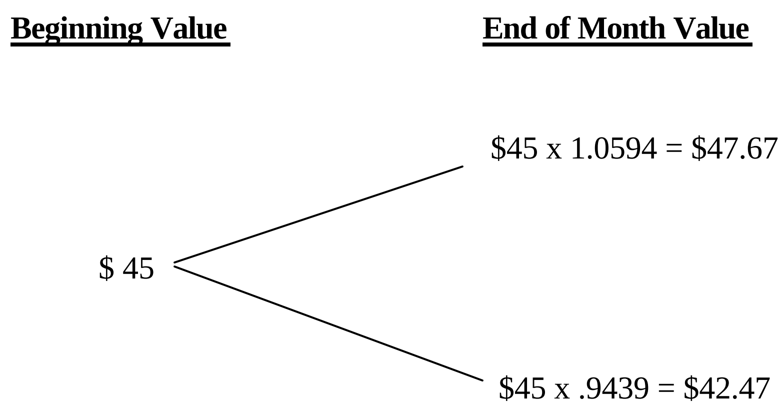
This is the same value we got using the risk less hedge approach. It is not easy but it can be shown that while the approaches appear to be different, they are the same. As such, either approach can be used.

Estimating the Binomial Stock price processes.

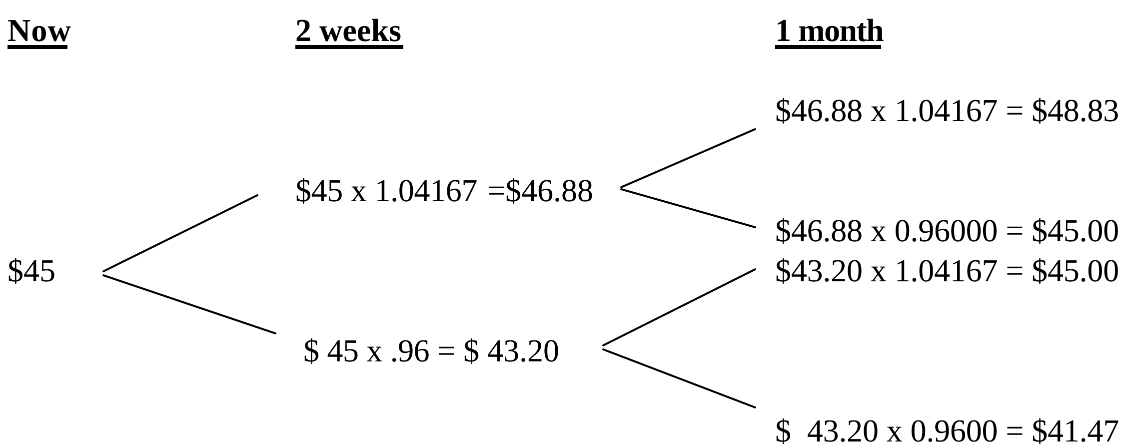
One of the difficulties encountered in implementing the binomial model is the need to specify the stock price process in a binomial tree. While it is not transparent, when we use the Black-Scholes model we are assuming a very explicit functional form for the stock price. If we are willing to make the same assumptions when we are using the binomial model we can construct a binomial model of the price process by using the volatility, s, to estimate up, ***u***, and down, ***d***, price movements[3]. This is done in practice as

$$\boldsymbol{u} = e^{s\sqrt{\Delta t}} \text{ and } \boldsymbol{d} = e^{-s\sqrt{\Delta t}},$$

where s is annual volatility and Δt is the time between price changes. For example, assume a current stock price of $55, a volatility of .20, s = .20 and that the time to between price changes is 1 month, Δt = .08333. Then $\boldsymbol{u} = e^{s\cdot\sqrt{\Delta t}} = e^{.20\cdot\sqrt{.08333}}$ = 1.0594 and then $\boldsymbol{d} = e^{-s\sqrt{\Delta t}} = e^{-.20\cdot\sqrt{.08333}}$ = 0.9439 and the stock price process over the one month interval would be:



If we keep the same ending point but let the price change every 2 weeks (Δt=.04167), then $\boldsymbol{u} = e^{s\cdot\sqrt{\Delta t}} = e^{.20\cdot\sqrt{.04167}}$ = 1.04167 and $\boldsymbol{d} = e^{-s\cdot\sqrt{\Delta t}} = e^{-.20\cdot\sqrt{.04167}}$ = 0.96 and the stock price process over the one month interval would be:



---

3 For a complete discussion of the assumptions underlying this approach, see Rendleman, R. and B. Bartter, "Two State Option Pricing," *Journal of Finance*, 34 (1979), 1092-1110.

In the limit we could allow the price change for the example used above to change every week (four times during the month and $\Delta t = 1/52 = .01923$) or daily (twenty-one times during the month[4] and $\Delta t = 1/252 = .00397$). Thus, given a volatility estimate we can construct the price process for that security. Once the price process for the underlying security is determined it is possible to use the binomial model to price options on that security.

Summary

Given the emphasis on the Black-Scholes model it may seem strange but most sophisticated option pricing models use some form of the Binomial Option Pricing model and not the Black-Scholes model. The approach is to estimate the price process of the underlying asset over the maturity of the option and then overlay the option payoffs given the values of the underlying asset. Pricing is done on the same basis as presented above. The primary reason that the Binomial model is used is its flexibility compared to the Black-Scholes model. It can be used to price a wide variety of options.

[4] When we get to using daily price changes, the general approach is to use trading days during a year or time period and not calendar days.